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APPLICATION OF STRUCTURAL REPRESENTATIONS TO THE SOLUTION OF BOUNDARY-VALUE PROBLEMS OF IDEAL PLASTICITY*

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The method of successive approximations is proposed for solving plane problems of the theory of ideal plasticity, based on structural representations. By using this method the state of stress is determined in an infinite plane with a circular hole for an arbitrary change in the forces applied at infinity. It is shown that for a certain constraint on the asymmetry of the loads, the solution of the problem considered is independent of the loading trajectory. When the mentioned constraint is violated, the plasticity domain will be different depending on the history of the load change.

1. The stresses caused by inelastic strain $\Gamma_{ij}\ (i, j = x, y, z)$ can be represented in the plane case /1/ as stresses due to wedgelike dislocations (WD), distributed (inserted) over the plasticity domain (PD) with density $p(x, y)$ and over its boundary L with density $p_L(l)$. The magnitudes of the densities under plane strain have the form

$$p(x, y) = \frac{\partial^2\Gamma_x}{\partial y^2} + \frac{\partial^2\Gamma_y}{\partial x^2} - 2\frac{\partial^2\Gamma_{xy}}{\partial x\partial y} + \nu\Delta\Gamma_z \tag{1.1}$$
$$p_L(l) = \left(\frac{\partial\Gamma_{xy}}{\partial y} - \frac{\partial\Gamma_y}{\partial x}\right)\cos(nx) + \left(\frac{\partial\Gamma_{xy}}{\partial x} - \frac{\partial\Gamma_x}{\partial y}\right)\cos(ny) - \nu\frac{\partial\Gamma_z}{\partial n}$$

($\Gamma_i = \Gamma_{ii}$, ν is Poisson's ratio, and n is the external normal to the boundary of the plasticity domain).

In the plane state of stress the values of the densities are obtained from the expression presented if the last terms dependent on the strain Γ_z are discarded.

On the other hand, if the state of stress of a body is known, it completely determines those structural imperfections that were formed therein at the time under consideration. The density of these imperfections (i.e., the WD) in the PD and on its boundary is determined by the expressions /1/

$$p(x, y) = -\frac{1+\varkappa}{8G}\Delta(\sigma_x + \sigma_y) \tag{1.2}$$

$$p_L = -\frac{1+\varkappa}{8G}\left[\frac{\partial(\sigma_x+\sigma_y)}{\partial n}\right]_L \tag{1.3}$$

The square brackets here denote discontinuities of the quantities therein on the inelastic strain domain boundary; it is calculated for the passage from points within the domain

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to points outside the domain; $\varkappa = 3 - 4\nu$ for plane strain and $\varkappa = (3-\nu)/(1+\nu)$ for the plane state of stress, where G is the shear modulus.

On the basis of the structural representations introduced, the method of successive approximations is proposed for solving statically determinate problems of ideal plasticity. Among them, for instance, are the generalized problems of L.A. Galin and G.P. Cherepanov on the strain of an infinite plane with a circular hole when the PD cannot completely enclose the hole.

In the problems under consideration the stress components and the WD density in the PD are known (they will be the same as in the solutions /2, 3/). It is required to find the boundary separating the elastic and plastic domains and also the state of stress in the elastic domain.

To a first approximation the stress field outside the PD is taken from the elastic solution. The approximate location of the PD boundary is determined on the basis of the maximum shear stress being equal to the yield point. Afterwards, the approximate value of the WD density on this boundary is found from the elastic solution. In general, other structural imperfections (edge dislocations) distributed over the boundary between the domains are also formed, in addition to the WD, at each step of the approximation. However, because of the solution of the problem a continuous stress field should be obtained and it can be caused only by WD. Consequently, other components of the structural imperfections are discarded. The state of stress is calculated in the refined elastic domain from the WD density found. The cycle is then repeated.

2. We will examine the application of the method of successive approximations elucidated in the example of solving the generalized Galin problem. In this case the state of stress in the PD that always adjoins the contour of a hole of radius R has the form (τ is the yield point)

$$\sigma_r{}^q + \sigma_\theta{}^q = 2\tau\left(1 + 2\ln\frac{r}{R}\right), \quad \sigma_\theta{}^q - \sigma_r{}^q = 2\tau, \quad \tau_{r\theta}^q = 0 \tag{2.1}$$

The WD density here equals zero in the inelastic strain domain.

The problem is to determine the PD domain and the WD density p_L distributed over the boundary. For a known density p_L the determination of the state of stress and strain reduces to the evaluation of quadratures.

As already mentioned, in a first approximation the state of stress outside the PD is taken from the elastic solution

$$\sigma_r^{(1)} = \sigma_r{}^y, \quad \sigma_\theta^{(1)} = \sigma_\theta{}^y, \quad \tau_{r\theta}^{(1)} = \tau_{r\theta}^y \tag{2.2}$$

Then the PD boundary is determined from the condition that the maximum shear stress equals the yield point (τ).

Knowing the state of stress and the location of the line L, the density WD on the PD boundary can be calculated from (1.3). We obtain

$$p_{L1} = -\frac{1-\nu}{2G}\left[\frac{\partial(\sigma_\theta^{(1)} + \sigma_r^{(1)})}{\partial n} - \frac{\partial(\sigma_\theta{}^q + \sigma_r{}^q)}{\partial n}\right] \tag{2.3}$$

Besides these structural imperfections, the WD are also distributed on the circular boundary of the body. Taking into account that the PD boundary agrees in this case with the boundary of the body, the density of the structural imperfections thereon is found on the basis of expression (1.3)

$$p_R = \frac{1-\nu}{2G}\frac{\partial(\sigma_\theta{}^q + \sigma_r{}^q)}{\partial n}\bigg|_{r=R} = -\frac{2\tau(1-\nu)}{GR} \tag{2.4}$$

To determine the state of stress caused by structural imperfections, we evaluate the stress function from two unit WD located at the points z_0 and $-z_0$ of an infinite plane with a circular hole. Using the stress function from one WD /4/, we obtain

$$\Phi(z) = \frac{G}{4\pi(1-\nu)}\left[\ln\frac{(z^2\bar{z}_0{}^2 - R^4)R^2}{z^2\bar{z}_0{}^2(z^2 - z_0{}^2)} - \frac{2R^2(R^2 - r_0{}^2)}{R^4 - z^2\bar{z}_0{}^2}\right]$$
$$\Psi(z) = \frac{G}{4\pi(1-\nu)}\left[\frac{2R^4(R^4 r_0{}^2 + 2z^2\bar{z}_0{}^2R^2 - 3z^2\bar{z}_0{}^2r_0{}^2)}{z^2(R^4 - z^2\bar{z}_0{}^2)^2} + \frac{2r_0{}^2}{z^2 - z_0{}^2} - \frac{2R^2}{z^2}\ln\frac{r_0{}^2}{R^2}\right]$$

On the basis of the stress functions presented we determine the stress components from the WD (2.4) located on a part of the circular boundary

$$\sigma_\theta^R + \sigma_r^R = \int_{-\theta_*}^{\theta_*} (\sigma_\theta^d + \sigma_r^d)\, p_R R\, d\theta$$

where θ_* is the angle characterizing the point of intersection of the boundary L with the hole contour $\sigma_\theta^d + \sigma_r^d = 4\,\mathrm{Re}\,\Phi(z)$.

The two other itegrals are written analogously. After evaluation of the integrals we obtain

$$\sigma_\theta^R + \sigma_r^R = \frac{8\theta_*\tau}{\pi} \ln \frac{r}{R}$$

$$\sigma_\theta^R - \sigma_r^R = \frac{4\theta_*\tau}{\pi}\left(1 - \frac{R^2}{r^2}\right), \quad \tau_{r\theta}^R = 0$$

The last components in all stages of the approximation will have the same form since the stress field in the PD does not change.

Taking into account that the state of stress in a real body can be represented as sum of actions on an elastic body by an external load and structural imperfections, we find the stress components in the elastic domain in the next step of the approximation

$$\sigma_\theta^{(2)} + \sigma_r^{(2)} = \int_L (\sigma_\theta^d + \sigma_r^d)\, p_{L1}\, dl + \sigma_\theta^R + \sigma_r^R + \sigma_\theta^q + \sigma_r^q \qquad (2.5)$$

The components $\sigma_\theta^{(2)} - \sigma_r^{(2)}$ and $\tau_{r\theta}^{(2)}$ can be written in the same manner.

The cycle is later repeated.

The approximation process is continued until the greatest difference $|\Delta r_0|$ between the last two approximations of the radius-vector of the boundary L becomes less than a given quantity ($|\Delta r_0| \leqslant 0{,}1R$).

3. By using this method of approximate solution, two kinds of loading were investigated for the generalized Galin problem. Solutions for three values of the external forces were obtained in each loading program.

The first approximation was found from the elastic solution in the initial loading stage. The solution obtained in the preceding loading stage was taken as the first approximation for succeeding values of the load.

In order to achieve given accuracy ($|\Delta r_0| \leqslant 0{,}1\,R$) at each loading stage, it is sufficient to make three approximations.

Let us first examine proportional loading when the load ratio $\beta_* = q_x/q_y$ does not vary. We set $\beta_* = 0{,}75$. The plastic strains then occur for $q_x \geqslant 0{,}59\tau$, $q_y \geqslant 0{,}89\tau$.

The diagram of PD development as the load increases is shown in Fig.1 for $q_x = 1{,}05\tau$ $q_y = 1{,}4\tau$ (curve *1*); $q_x = 1{,}2\tau$, $q_y = 1{,}6\tau$ (curve *2*); $q_x = 1{,}5\tau$, $q_y = 2\tau$ (curve *3*). Graphs of the appropriate values of the WD density are shown in Fig.2.

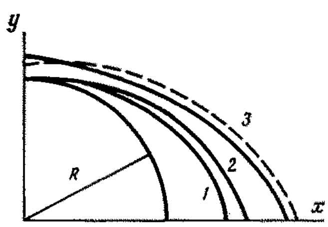

Fig.1

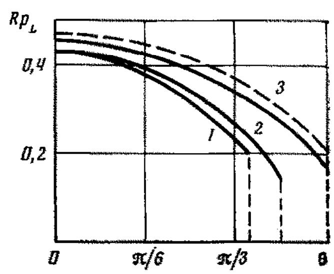

Fig.2

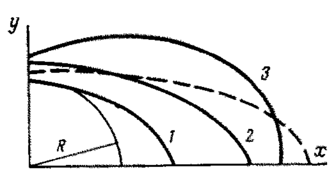

Fig.3

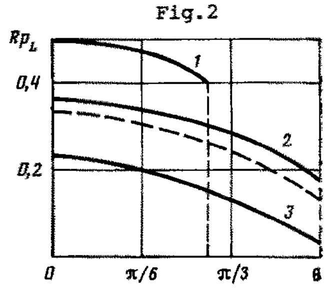

Fig.4

Under the external forces $q_x = 1{,}5\tau$, $q_y = 2\tau$ the PD boundary obtained from the Galin solution encloses the circular hole. It is shown by dashes in Fig.1. Values of p_L corresponding to the solution mentioned are given by dashes in Fig.2.

It is seen that the Galin solution of the problem and the method of successive approximations are practically in agreement for load values $q_x = 1{,}5\tau$, $q_y = 2\tau$. This means that when the conditions of monotonicity of the PD development and the constraint on load asymmetry are satisfied

$$q_y - q_x \leqslant 0{,}82\tau \tag{3.1}$$

the solution obtained by Galin also corresponds to those loading trajectories for which incomplete enclosure of the circular hole by the PD is possible during strain. Under the conditions mentioned, the Galin solution is obviously independent of the history of the change in external forces.

We consider the loading program $(q_x/\tau;\ q_y/\tau) = (1{,}2;\ 2),\ (1{,}9;\ 2{,}6),\ (2;\ 3)$, terminated by forces for which the condition of PD enclosure of the circular hole is satisfied but the condition imposing the constraint (3.1) on the load asymmetry is spoiled. The PD and the values of the WD density are shown in Figs.3 and 4 for the load $q_x = 1{,}2\tau, q_y = 2\tau$ (curve *1*); $q_x = 1{,}9\tau, q_y = 2{,}6\tau$ (curve *2*), $q_x = 2\tau, q_y = 3\tau$ (curve *3*).

The PD boundary and the graph of WD density values obtained on the basis of the Galin solution for $q_x = 2\tau$, $q_y = 3\tau$ are shown by dashes.

It follows from the example presented that if the load asymmetry does not safisfy condition (3.1), then the state of stress and strain of a plane with a circular hole depends on the loading trajectory (the history of the change in external forces).

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ELASTIC EQUILIBRIUM OF CIRCULAR PIECEWISE-HOMOGENEOUS MEDIA WITH A DIAMETRAL CRACK*

A.I. SOLOV'YEV

A method is proposed for solving boundary value problems of elasticity theory for circular piecewise-homogeneous media with a symmetric diametral crack, based on the application of vector relationships between the basis solutions of the equilibrium equations in polar and elliptic coordinates. The realization of this method results in infinite systems of linear algebraic equations of the second kind with exponentially decreasing matrix coefficients.

The problem of the equilibrium of a two-component piecewise-homogeneous plane with a symmetric diametral crack is considered in an inner homogeneous domain. Asymptotic formulas for the stress intensity coefficients are obtained by expansion in a small parameter.

1. The elliptic coordinates are related to the Cartesian coordinates by the formulas

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